# S620 - Introduction To Statistical Theory - Homework 6

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1. Show that the Poisson distribution forms an exponential family.

   **Solution:** Let $X \sim Pois(\theta)$. The pmf of $X$ is $f_X(x;\theta) = e^{-\theta}\frac{\theta^x}{x!}$. We want to show that this pmf can be written as $f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\}$. For a Poisson distribution, let $c(\theta) = e^{-\theta}, h(x) = \frac{1}{x!}, k = 1$ and $\pi_1(\theta) = log(\theta), \tau_1(x) = x$. Then:

   $$f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\} = c(\theta)h(x)exp\{\pi_1(\theta)\tau_1(x)\} = e^{-\theta}\frac{1}{x!}exp\{xlog(\theta)\} = e^{-\theta}\frac{\theta^x}{x!} = f_X(x;\theta)$$

   Showing that the Poisson distribution forms an exponential family.

2. Show that the Binomial distribution forms an exponential family.

   **Solution:** Let $X \sim Bin(n,\theta)$, for known $n$ and unknown $\theta$. The pmf of $X$ is $f_X(x;\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}$. We want to show that this pmf can be written as $f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\}$. For a Binomial distribution, let $c(\theta) = 1, h(x) = \binom{n}{x}, k = 2$ and $\pi_1(\theta) = log(\theta), \tau_1(x) = x; \pi_2(\theta) = log(1-\theta), \tau_2(x) = n - x$. Then:

   $$f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\} = c(\theta)h(x)exp\{\pi_1(\theta)\tau_1(x)+\pi_2(\theta)\tau_2(x)\} = 1\cdot\binom{n}{x}\cdot exp\{xlog(\theta)+(n-x)log(1-\theta)\}$$

   $$= \binom{n}{x}exp\{log(\theta^x)\}exp\{log\left[(1-\theta)^{n-x}\right]\} = \binom{n}{x}\theta^x(1-\theta)^{n-x} = f_X(x;\theta)$$

   Showing that the Binomial distribution forms an exponential family.

3. Show that the Geometric distribution forms an exponential family.

   **Solution:** Let $X \sim Geo(\theta)$. The pmf of $X$ is $f_X(x;\theta) = \theta(1-\theta)^{x-1}$. We want to show that this pmf can be written as $f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\}$. For a Geometric distribution, let $c(\theta) = \theta, h(x) = 1, k = 1$ and $\pi_1(\theta) = log(1-\theta), \tau_1(x) = x - 1$. Then:

   $$f(x;\theta) = c(\theta)h(x)exp\{\sum_{i=1}^{k}\pi_i(\theta)\tau_i(x)\} = c(\theta)h(x)exp\{\pi_1(\theta)\tau_1(x)\} = \theta\cdot 1\cdot exp\{(x-1)log(1-\theta)\} = \theta(1-\theta)^{x-1} = f_X(x;\theta)$$

   Showing that the Geometric distribution forms an exponential family.

For the next three problems, we want to show that $(G,\circ)$ is a group, where $G = \{g_\theta : \theta \in \Theta = \mathbb{R}^p\}$, such that $g_\theta : \mathbb{R}^p \to \mathbb{R}^p$, and $\circ$ denotes function composition. The particular action of $g_\theta$ is different in each case.

4. Show that location families form a transformation group.

   **Solution:** Let $g_\theta \in G$ be defined as $g_\theta(x) = x + \theta$. Let us check all properties of a group:

   (a) *Closure:* Let $g_{\theta_1}, g_{\theta_2} \in G$. Then,

   $$(g_{\theta_1} \circ g_{\theta_2})(x) = g_{\theta_1}(g_{\theta_2}(x)) = g_{\theta_1}(x + \theta_2) = (x + \theta_2) + \theta_1 = x + (\theta_1 + \theta_2) = g_{\theta_1+\theta_2}(x)$$

   This is of the correct form, i.e., $g_{\theta_1+\theta_2} \in G$, and so $\circ$ is a binary operation on $G$.

   (b) *Associativity:* In general, composition of functions from a set to itself is an associative operation. In this case, let $g_{\theta_1}, g_{\theta_2}, g_{\theta_3} \in G$. Then:

$g_{\theta_1} \circ (g_{\theta_2} \circ g_{\theta_3}) = (g_{\theta_1} \circ (g_{\theta_2} \circ g_{\theta_3}))(x) = g_{\theta_1}(g_{\theta_2}(g_{\theta_3}(x))) = g_{\theta_1}(g_{\theta_2}(x + \theta_3)) = g_{\theta_1}(x + \theta_3 + \theta_2) = x + \theta_3 + \theta_2 + \theta_1 = ((x + \theta_3) + \theta_2) + \theta_1$
$= g_{\theta_1}(g_{\theta_2}(g_{\theta_3}(x))) = ((g_{\theta_1} \circ g_{\theta_2}) \circ g_{\theta_3})(x) = (g_{\theta_1} \circ g_{\theta_2}) \circ g_{\theta_3}$, thus, the binary operation is associative on $G$.

(c) *Existence of identity element:* Let us check that $g_{\vec{0}} \in G$ is the identity element. Let $g_\theta \in G$. Then:

$$(g_\theta \circ g_{\vec{0}})(x) = g_\theta(g_{\vec{0}}(x)) = g_\theta(x+0) = g_\theta(x) = x+\theta = (x+\theta)+0 = g_{\vec{0}}(x+\theta) = g_{\vec{0}}(g_\theta(x)) = (g_{\vec{0}} \circ g_\theta)(x)$$

Hence, $g_\theta \circ g_{\vec{0}} = g_{\vec{0}} \circ g_\theta = g_\theta$.

(d) *Existence of inverse elements:* Let $g_\theta \in G$ be given. Let us check that $g_{-\theta} \in G$ is its inverse:

$$(g_\theta \circ g_{-\theta})(x) = g_\theta(g_{-\theta}(x)) = g_\theta(x+(-\theta)) = (x+(-\theta))+\theta = x+0 = g_{\vec{0}} = (x+\theta)+(-\theta) = g_{-\theta}(x+\theta) = g_{-\theta}(g_\theta(x)) = (g_{-\theta} \circ g_\theta)(x)$$

Hence, $g_\theta \circ g_{-\theta} = g_{-\theta} \circ g_\theta = g_{\vec{0}}$

5. Show that scale families form a transformation group.

**Solution:** Let $g_\theta \in G$ be defined as $g_\theta(x) = \theta x$ [in this case $\Theta = (0,\infty)$]. Let us check:

(a) *Closure:* Let $g_{\theta_1}, g_{\theta_2} \in G$. Then,

$$(g_{\theta_1} \circ g_{\theta_2})(x) = g_{\theta_1}(g_{\theta_2}(x)) = g_{\theta_1}(\theta_2 x) = \theta_1(\theta_2 x) = (\theta_1\theta_2)x = g_{\theta_1\cdot\theta_2}(x)$$

This is of the correct form, i.e., $g_{\theta_1\cdot\theta_2} \in G$, and so $\circ$ is a binary operation on $G$.

(b) *Associativity:* The argument is essentially the same as in 4.

(c) *Existence of identity element:* Let us check that $g_1 \in G$ is the identity element. Let $g_\theta \in G$. Then:

$$(g_\theta \circ g_1)(x) = g_\theta(g_1(x)) = g_\theta(1\cdot x) = g_\theta(x) = \theta x = 1\cdot(\theta x) = g_1(\theta x) = g_1(g_\theta(x)) = (g_1 \circ g_\theta)(x)$$

Hence, $g_\theta \circ g_1 = g_1 \circ g_\theta = g_\theta$.

(d) *Existence of inverse elements:* Let $g_\theta \in G$ be given. Let us check that $g_{\frac{1}{\theta}} \in G$ is its inverse:

$$(g_\theta \circ g_{\frac{1}{\theta}})(x) = g_\theta(g_{\frac{1}{\theta}}(x)) = g_\theta(\frac{1}{\theta}x) = \theta(\frac{1}{\theta}x) = 1\cdot x = g_1(x) = \frac{1}{\theta}(\theta x) = g_{\frac{1}{\theta}}(\theta x) = g_{\frac{1}{\theta}}(g_\theta(x)) = (g_{\frac{1}{\theta}} \circ g_\theta)(x)$$

Hence, $g_\theta \circ g_{\frac{1}{\theta}} = g_{\frac{1}{\theta}} \circ g_\theta = g_1$

6. Show that location-scale families form a transformation group.

**Solution:** Let $g_\theta \in G$ be defined as $g_\theta(x) = \eta + \tau x$ [in this case $g_\theta : \mathbb{R} \to \mathbb{R}$, and $\Theta = (-\infty,\infty)\times(0,\infty)$, and thus, $\theta = (\eta,\tau)$]. Then:

(a) *Closure:* Let $g_{\theta_1}, g_{\theta_2} \in G$, where $\theta_1 = (\eta_1,\tau_1)$ and $\theta_2 = (\eta_2,\tau_2)$. Then,

$$(g_{\theta_1} \circ g_{\theta_2})(x) = g_{\theta_1}(g_{\theta_2}(x)) = g_{\theta_1}(\eta_2+\tau_2 x) = \eta_1+\tau_1(\eta_2+\tau_2 x) = \eta_1+\tau_1\eta_2+\tau_1\tau_2 x = (\eta_1+\tau_1\eta_2)+(\tau_1\tau_2)x = g_\theta(x)$$

This is of the correct form, i.e., $g_\theta \in G$, where $\theta = (\eta_1+\tau_1\eta_2, \tau_1\tau_2)$ and so $\circ$ is a binary operation on $G$. Note that $\tau_1\tau_2 \in (0,\infty)$ since both $\tau_1$ and $\tau_2$ are in $(0,\infty)$

(b) *Associativity:* The argument is essentially the same as in 4.

(c) *Existence of identity element:* Let us check that $g_e \in G$, where $e = (0,1)$ is the identity element. Let $g_\theta \in G$. Then:

$$(g_\theta \circ g_e)(x) = g_\theta(g_e(x)) = g_\theta(0+1\cdot x) = g_\theta(x) = \eta+\tau x = 0+1\cdot(\eta+\tau x) = g_e(\eta+\tau x) = g_e(g_\theta(x)) = (g_e \circ g_\theta)(x)$$

Hence, $g_\theta \circ g_e = g_e \circ g_\theta = g_\theta$.

(d) *Existence of inverse elements:* Let $g_\theta \in G$ be given, where $\theta = (\eta,\tau)$. Let us check that $g_{\theta^{-1}} \in G$, where $\theta^{-1} = (-\frac{\eta}{\tau}, \frac{1}{\tau})$, is its inverse. [Note that division by $\tau$ is well defined because $\tau \in (0,\infty)$]. Then:

$$(g_\theta \circ g_{\theta^{-1}})(x) = g_\theta(g_{\theta^{-1}}(x)) = g_\theta(-\frac{\eta}{\tau}+\frac{1}{\tau}x) = \eta+\tau(-\frac{\eta}{\tau}+\frac{1}{\tau}x) = \eta+(-\eta)+\frac{\tau}{\tau}x = 0+1\cdot x = g_e(x)$$

$$(g_{\theta^{-1}} \circ g_\theta)(x) = g_{\theta^{-1}}(g_\theta(x)) = g_{\theta^{-1}}(\eta+\tau x) = (-\frac{\eta}{\tau})+\frac{1}{\tau}(\eta+\tau x) = (-\frac{\eta}{\tau}+\frac{\eta}{\tau})+\frac{\tau}{\tau}x = 0+1\cdot x = g_e(x)$$

Hence, $g_\theta \circ g_{\theta^{-1}} = g_{\theta^{-1}} \circ g_\theta = g_e$